Price Determination in RBS

Note – Below appears excerpts from an article published about The ChicagoGSB Registration Bidding System, titled "An Auction Method for Course Registration". This article was written by Robert L. Graves, Graduate School of Business, University of Chicago; Linus Schrage, Graduate School of Business, University of Chicago; and Jayaram Sankaran, Department of Management Studies, Indian Institute of Science. The article appeared in *Interfaces* we:5 September-October 1993 (pp. 81-92).

[As outlined in the ORB instructions] The current system includes the following features. First, it captures the preferences of students for course sections. Second, it enables students to express contingency constraints on the course sections for which they register. Third, it allows students whose preferences or needs change to rebid in a sequence of auctions before and after the term begins. Fourth, it is designed for a large problem.

Recall that the highest successful bid for each bidder is awarded and subsequent bids for this bidder are ignored. Both successful bids and all unsuccessful bids are retained as activities in the linear program used for determining prices. Here are the definitions of the parameters and decision variables:

B_{ij}= bid of student *i* for schedule *j*,

$A_{ijk} =$ 1 if schedule *j* includes course *k*, and else 0,

C_k = capacity of course *k*, and

X_{ij} = fraction of schedule *j* assigned to student *i*.

The essential elements are captured in the following linear program

Max $\sum_i å_j b_{ij} X_{ij}$: total utility

Subject to

1) $\sum_i \sum_j a_{ijk} X_{ij} \leq C_k$:section *k* capacity (*m* rows)

2) $\sum_j X_{ij} \leq 1$:student *i* capacity (*n* rows)

we require a solution in which each X_{ij} is 0

The most natural "true" solution is to solve this problem as an integer program with all schedules considered and with the relations1) and 2) exactly as stated. We do not consider all schedules submitted by the students, but only the subset that are successful or unsuccessful. Next we modify the relations replacing each inequality in 2) by an equality for each student who received a schedule in the queuing process. This modified linear program ensures that each schedule selected in the queuing process is part of a primal feasible solution. Specifically, the X_{ij} associated with awarded schedules are set to 1; the X_{ij} associated with rejected schedules are set to 0. The primal solution is integral, unique, and perforce must be optimal. (The primal basis is not necessarily unique.) The linear program is not used to obtain the primal solution (which is available from the queuing process) but rather to obtain the dual solution which gives prices for the closed course sections.

An example with the following objective (as row 1 of an LP) illustrates the main ideas:
Max = 40A1 + 36A2 + 32A3 + 28A4 + 25A5 + 20A6 + 32B1 + 38B2 + 27B3+ 33B4 + 24B5 + 20B6 + 33C1 + 33C2 + 36C3 + 29C4 + 14C5 +14C6 + 28D1 + 26D2 + 35D3 + 34D4 + 14D5 + 10D6 + 25E1 + 35E2 + 11E3 + 10E4 + 33E5 + 32E6 + 32F1 + 31F2 + 26F3 + 15F4 + 14F5 + 13F6.

Six students (A, B, C, D, E, F) each wish to register for a schedule of two courses from a total of four courses (2, 3, 4, 5). The course labels correspond to rows 2 through 5 whose RHS – 3, 2, 3, 3 – are the capacities of corresponding courses. This means that a course schedule is one of the six pairs: (2,3), (2,4), (2,5), (3,4), (3,5), or (4,5). These schedules are designated by the numbers 1,2, 3, 4, 5, and 6. In the LP, A1 = the fraction of schedule 1 awarded to A and F6 = the fraction of schedule 6 awarded to F. we require a solution in which each variable is either 0 or 1.

The coefficients in the objective function are the amounts each student is willing to pay for the corresponding schedule. For example, A is willing to pay (at most) 40 for schedule 1 but is willing to pay at most 20 for schedule 6. Since the total number of sears is 11, it is clear that not all six students can get a schedule because 12 seats would be required. The RHS for constraints 6 through 11 ensures that the total of the (possibly) fractional schedules awarded to each student does not exceed 1.

One might try to solve the "true" problem by rounding LP solutions or by using IP. When RBS was developed, there was little prospect that either of these approaches would be practical in routine use. The results of Sankarn (1989) suggest that this may be changing. Our approach yields these activities for the LP.
Feasible – A1, B2, C3, D4, E6
Infeasible – D3, E2, E5, F1, F2, F3, F4, F5, F6

The optimal objective of solution to the original (continuous) LP which contains all 36 schedules in 187. The solution contains fractional values for several of the variables. The value of the objective for the substitute problem is 180. Part of the loss occurs because the integer solution requires that one space be left empty in course 5.

EACH YEAR, IT ASSIGNS 18,000 SEATS IN ABOUT 475 COURSE SECTIONS.

The values of the dual prices are determined by the objective coefficients (bids) of the primal basic variables; in any LP the dual prices of the resources are determined by market prices through the technological choices that maximize total return. Here the rules are bent by eliminating some of the choices eliminated are the schedules less preferred than the schedule actually awarded and the slack variables of those bidders who do get a schedule. Course prices are actually determined by the basic schedules that have a value of 0. These correspond to rejected schedules that are almost awarded and are analogous to the highest unsuccessful bid in an auction for one type of item, a so-called Vickrey auction (1961).

MAX = 0D2A + 13A5A + 15S42D + 17S45D + 12S52E + 17S53E + 20C2F + 21C3F + 18C4F + 16C5F; Objective function showing student bids

-D2A + S42D+S52E+C2F	≤0.01;	Course 2 balance
+S53E + C3F	≤0.001;	Course 3 balance
-S42D-S45D+C4F	≤0.0001;	Course 4 balance
+A5A + S45D –S52E+ C5F	≤1.00001;	Course 5 balance
D2A ≤ 1;		Student limits on
A5A ≤ 1;		adding or dropping
S42D-S45D≤1;		courses
S52E + S53E ≤1;		
C2F + C3F ≤ 1;		
C4F + C5F ≤ 1;		

END

VARIABLE	VALUE	REDUCED COST
D2A	1.000000	.000000
A5A	1.000000	.000000
S42D	.0000000	14.000000
S45D	1.000000	.000000
S52E	.9989900	.000000
S53E	0.00100000	.000000
C2F	0.01101001	.000000
C3F	.0000000	4.000000
C4F	1.000000	.000000
C5F	.0000000	10.000000

The variables which are approximately 1 correspond to successful bids.

ROW	SLACK OR SURPLUS	DUAL PRICE
1	60.22508	1.00000
2	.0000000	20.00000
3	.0000000	25.00000
4	0.0001000	.00000
5	.0000000	8.00000
6	.0000000	20.00000
7	.0000000	5.00000
8	.0000000	9.00000

9	0.00001001	.00000
10	.9889900	.00000
11	.0000000	18.00000

The dual prices of rows 2 through 5 are the prices of the courses.

Table 2: Solution of drop/add/swap example

Denote the dual prices of the course capacity constraints by $P_2, \ldots, P_5$ and the dual prices of the student capacity constraints by $P_A, \ldots, P_F$. Then the dual relations associated with the variables shown in parentheses determine the prices of the courses:

$P_5 = 0$ (slack course 5),
$P_F = 0$ (slack student F)
$P_4 + P_5 + P_F = 13$(F6) :$P_4 = 13$,
$P_4 + P_5 + P_E = 32$(E6) :$P_E = 19$,
$P_3 + P_5 + P_E = 33$(E5) :$P_3 = 14$,
$P_3 + P_4 + P_D = 34$(D4) :$P_D = 7$, and
$P_2 + P_5 + P_D = 35$(D3) :$P_2 = 28$

The dual prices of the student capacities represent the consumer surplus for each student. For most students, the total actual payment for the schedule is the amount bid less the consumer surplus which equals the total value of the courses in the schedule. An anomalous (but expected) situation arises when the surplus is negative. In actual use of the system, the student pays the smaller of the value of the bid and the total value of the courses. In this example,

A pays min (28 + 14, 40) = 40
:subsidy = 2,

B pays min (28 + 13, 38) = 38
:subsidy = 3,
C pays min (28 + 0, 36) = 28
:subsidy = 0,

D pays min (14 +13, 34) = 27
:subsidy = 0

E pays min (13 + 0, 32) = 13
:subsidy = 0, and

F pays 0 :subsidy = 0.

Subsequent Markets

The RBS market does not guarantee that everyone will receive a schedule; further, a certain amount of shopping for courses occurs so that a follow-on market is necessary. The mechanism of drop-add-swap from the user's viewpoint has already been described. Here is an example of how allocations and prices are determined simultaneously.

Suppose that the bidders in the example wish to make these changes. The one or two associated variables used in the LP model are shown in parentheses. The bid amounts appear in the objective function of the model.

Student A:	Drop course 2 (D2A),
Student A:	Add course 5 (A5A),
Student B:	No charge
Student C:	No charge
Student D:	Swap course 4 for either course 2 or course 5 (S42D and S45D),
Student E:	Swap course 5 for either course 2 or course 3 (S52E and S53E),
Student F:	Add either course 2 or course 3 (C2F and C3F), and
Student F:	Add either course 4 or course 5 (C4F and C5F).

Table 2 shows the LP and its solutions. The course capacities are perturbed to show explicitly the marginal basic in this market. The constraints ensure that the variables do not exceed 1. The network structure of this problem guarantees that it always has (perturbations aside) an integral optimal solution so that the values of the variables are 0 or 1. In DAS, the solution to the LP both assigns bidders to courses and determined the prices of the courses. The price in the DAS market have no necessary relation to the prices determined in the RBS market. As before, prices are determined by almost successful bids at the margin – basic variables with small fractional values in the perturbed problem. The relations are:

P2 = 20 (C2F),
P4 = 0 (slack course 4),
P2 – P5 = 12 (S52E) so that P5 = 8, and
P3 – P5 = 17 (S53E) so that P3 = 25.

The solution of these equations agrees with that displayed in Table 2. The net payments of the bidders are:

A:	-20 + 8 = -12 (Drop 2 and add 5),
D:	-0 + 8 = 8 (Drop 4 and add 5),
E:	+20 – 8 = 17 (Drop 5 and add 2), and
F:	+ 0 = 0 (Add 4).